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Optimal Thrust Vector Control for Vertical Launch of Tactical Missiles

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This paper studies how a Differential Dynamic Programming (DDP) method can be used to compute optimal thrust vector controls for vertically launched missiles. Furthermore, the results of this investigation substantiate the practicality of a technique to control the convergence to overcome its difficulties with DDP. The problem under consideration requires the missile to perform a high-angle-of-attack maneuver. This, together with the control deflection limitation, implies nonlinear system dynamics. The optimal trajectories computed represent the performance that is possible to achieve given certain physical missile properties. The relative importance of different missile constraints is thus possible to compute.

Nomenclature

C_{D_0}	= drag coefficient
C_{mq}	= pitch rate damping coefficient
$C_{N\alpha}$	= normal force coefficient
d	= missile diameter
g	= acceleration due to gravity
I_{sp}	= specific impulse of propellant
I_y	= missile lateral moment of inertia
l	= missile length
l_c	= control length
l_0	= stability length
M	= Mach number
m	= missile mass
p_s	= static pressure
q	= dynamic pressure
r	= radius of inertia
S	= reference area
T	= thrust vector level
V	= missile total velocity
v, w	= missile velocity in a body-fixed system
v_s	= speed of sound
x,y	= missile position in an inertial space
α	= angle of attack
δ	= angle of deflection of the thrust vector
θ	= pitch angle
ω_y	= pitch rate

Introduction

THE reason for the considerable distance between theory and practice and for the sparseness of reported papers in this area when solving realistic nonlinear optimal control problems depends mainly on the difficulty of establishing good convergence in the numerical iterative procedures. The key to the successful operation of the modified first-order Differential Dynamic Programming (DDP) algorithm utilized in our examples is the method for achieving rapid convergence while maintaining stability of the algorithm. In connection with calculating optimal strategies associated with realistic missile vs aircraft problems, a comparison[1] between DDP and two other methods—a first-order gradient and a neighboring extremal method—has shown that the DDP method is preferable.

A first-order algorithm based on differential dynamic programming[2] and a convergence-control parameter technique[3] constitutes the framework in a Fortran program package.[4,5] This easy-to-use program makes it easy to overcome the convergence difficulties.

In this paper, an idea to differentiate the convergence control depending on the nonlinearity in the optimal control problem will be demonstrated.[5] In addition, the choice of nominal control to start the algorithm is not a difficult task. The choice of weight coefficients in the cost functional will also be discussed. The realistic problem in this paper, exemplified in Fig. 1, is concerned with computing optimal thrust vector controls for tactical missiles regarding the launch phase. Simulations of vertical launch of surface-to-air missiles have been reported but, to the best of our knowledge, the optimization problem has not been treated (at least not reported) previously.

Problem Background

Vertical launch of a tactical missile was performed at least as early as 1967 and since then has attracted interest in different studied system applications. Today, however, no tactical surface-to-air missile system is known to us to exploit vertical launch. Future systems may well possess this property. As an optimization problem, the vertical launch is possible to formulate very explicitly and possesses the properties of a nontrivial problem due to the nonlinearities of the aerodynamics and the control.

We must be aware of the fact that our formulation is far from complete and that the usefulness of a study in one plane is limited and does not exhibit all the characteristics of the full problem (e.g., out-of-plane moments and forces, induced rolling moments). The formulation is, however, believed to be accurate enough for computation of "optimal trajectories" in order to determine what is possible to achieve with a certain set of missile parameters and to determine the missile properties that will give a favorable vertical launch. The technique and problems discussed in the following apply to the problem of prior-to-launch computation of optimal reference trajectories (trajectory to be interpreted in a state-vector sense). The control of the missile relative to this reference trajectory is not treated.

Problem Formulation

Due to tactical reasons, vertical launch may be preferred in a missile system. Two of the optimization problems that may be formulated regarding vertical launch are:

1) Minimize the altitude at a given final time with the constraints that the missile body and the missile velocity vector are horizontal at this point.

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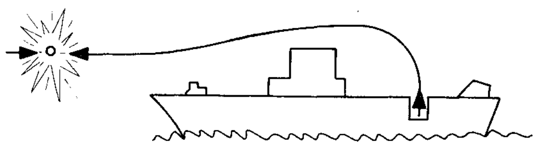

Fig. 1 Exemplified case study.

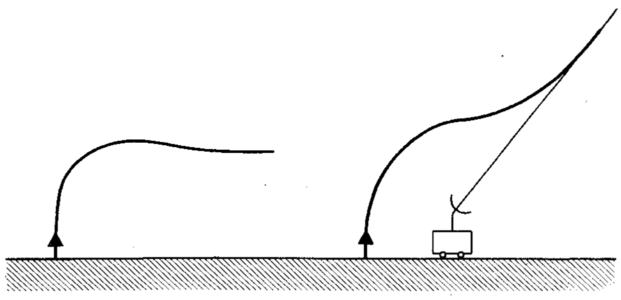

Fig. 2 Tactical requirements for interception of a target at low altitude (left) or positioning of a missile on a predicted line-of-sight (right).

2) Minimize the time to transfer the missile position from the initial position to a position on a given line with the x-axis body and the velocity vector parallel to this line.

These two cases, illustrated in Fig. 2, cover the intercept of a target at low altitude (left) and the positioning of a missile on the predicted line of sight between the on-ground equipment and the target (right). In a following section, these tactical requirements are mathematically formulated.

Mathematical Model

Using the notation given above and the coordinate frame shown in Fig. 3, the dynamics of the launch phase is given by Eqs. (1-3) in a body-fixed coordinate system:

$$\dot{w} = -\frac{q\cdot S}{m} C_N(\alpha) - \frac{T}{m}\cdot\sin(\delta) + \omega_y\cdot v + g\cdot\cos(\theta) \qquad (1)$$

$$\dot{v} = -\frac{q\cdot S}{m} C_{D_0} + \frac{T}{m}\cos(\delta) - \omega_y\cdot w - g\cdot\sin(\theta) \qquad (2)$$

$$\dot{\omega}_y = \frac{q\cdot S\cdot d}{I_y}\left(\frac{d}{2V}\cdot C_{mq}\cdot\omega_y - C_N(\alpha)\cdot\frac{l_0}{d}\right) - l_c\cdot\frac{T}{I_y}\cdot\sin(\delta) \qquad (3)$$

and

$$\alpha = \arcsin(w/V) = \arctan(w/v)$$

$$V^2 = v^2 + w^2 \qquad q = 0.7 p_s M^2 = 0.7(p_s/v_s^2)\cdot V^2$$

The position in an inertial frame is given by

$$\dot{x} = v\cos(\theta) + w\sin(\theta) \qquad \dot{y} = v\sin(\theta) - w\cos(\theta) \qquad (4)$$

$$\dot{\theta} = \omega_y \qquad (5)$$

$$\dot{m} = -T/I_{sp} \qquad (6)$$

Equation (6) reflects the fact that the missile mass is reduced when burning the boost propellant. The missile layout is given by Fig. 4 and the missile has the following properties. In order to get invariant results, we require:

$$(S/m)C_N(\alpha) = K_1 \qquad T/m = K_2 \qquad (S/m)C_{D_0} = K_3$$

$$S(d^2/I_y)C_{mq} = K_4 \qquad (Sl_0/I_y)C_N(\alpha) = K_5 \qquad l_c(T/I_y) = K_6$$

where K_i are constants. However, $C_N(\alpha)$, C_{D_0}, and C_{mq} are aerodynamic coefficients and are as such dependent of form only. Now the conditions for the missile fulfilling the results may be stated.

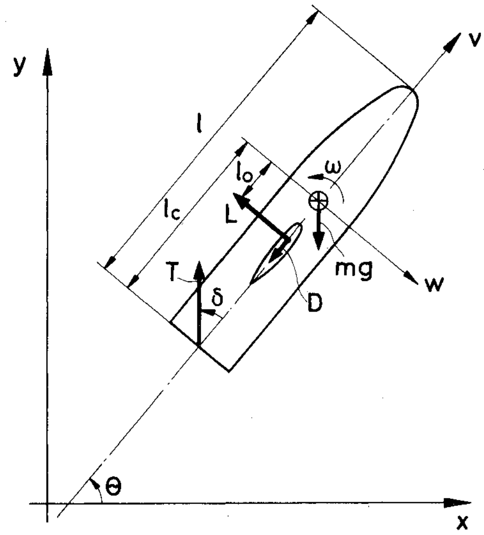

Fig. 3 Coordinate frame.

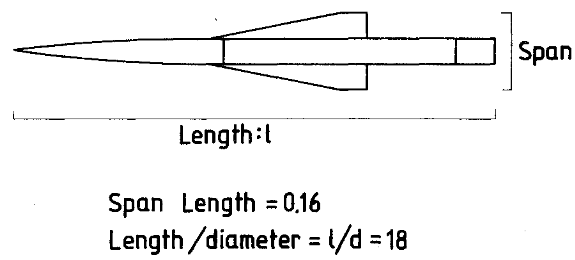

Fig. 4 Missile layout.

$$d/l = 1/18 \qquad S/m_0 = 1.30\cdot 10^{-4}\ \mathrm{m^2/kg}$$

$$T/m_0 = \text{parameter in results shown} \qquad m_{\text{burnout}} = 0.4\cdot m_0$$

$$l_0/r^2 = 0.42\ \mathrm{m^{-1}} \qquad l_c/r^2 = 4.17\ \mathrm{m^{-1}}$$

Aerodynamic coefficients are:

$$C_{D_0} = 0.2 \text{ for } M<1$$

$$= 0.6 \text{ for } M>1$$

$$C_N(\alpha) = 47.3\cdot\sin\alpha\cdot(0.5\cos\alpha + \sin\alpha) \qquad \text{for } M<1$$

$$= 27.6\cdot\alpha \qquad \text{for } M>1$$

$$C_{mq} = -5000$$

Thus, the model includes time-varying mass and moment of inertia. Furthermore, the saturation in normal force coefficient is modeled in the subsonic region. This is, of course, necessary in regard to expected large angles of attack.

The high value of C_{mq} has been chosen to simulate an inner rate gyro damping loop. From the start, this was considered necessary in order not to run into algorithm convergence difficulties caused by fuselage behavior, which is of comparatively little interest when the trajectory at large is optimized. However, different values of C_{mq} were used with no major differences in algorithm behavior, which is to be expected since the optimization results in open-loop trajectories and the control strategy that was arrived at compensates for the fuselage parameters. When implemented in a closed-loop form, certainly an inner rate gyro damping loop will be implemented.

The thrust level T and the maximum deflection angle δ_{max} were varied in the study to determine the influence of these parameters on the optimum trajectory.

Statement of the Optimal Control Problem

Consider the following nonlinear, minimizing control problem.

Minimize

$$V(x_0;t_0) = \int_{t_0}^{t_f} L(x,u;t)\,dt + F(x(t_f);t_f) \tag{7}$$

subject to

$$\dot{x} = f(x,u;t) \qquad x(t_0) = x_0 \tag{8}$$

$$g(x,u;t) \le 0 \tag{9}$$

$$h(x;t_f) = 0 \tag{10}$$

where x is an n-dimensional state vector and u is an m-dimensional control vector. The tactical requirements and the mathematical model given above will now be mathematically formulated in the considered optimization framework.

With the seven-dimensional state vector

$$x^T(t) = (x_1,\ldots,x_7) = (x,y,v,w,\theta,\omega,m) \tag{11}$$

and the scalar control variable

$$u(t) = \delta \tag{12}$$

the system of Eqs. (1-6) can be formulated as system (8). The constraint, Eq. (9), is given as

$$g(x,u;t) = |u| - u_{max} \le 0 \tag{13}$$

The first formulated tactical requirement is met if we choose

$$h(x;t_f) = \begin{pmatrix} x_5 \\ \frac{d}{dt} x_2 \end{pmatrix}_{t=t_f} = 0 \tag{14}$$

and

$$L(x,u;t) = 0 \qquad F(x(t_f);t_f) = p_1 \cdot x_2^2(t_f) \tag{15}$$

In this study, however, we have included the Eq. (14) requirement in the criterion functional, so we finally get Eq. (7) as

$$V(x_0;t_0) = F(x(t_f);t_f)$$

$$= p_1 \cdot x_2(t_f)^2 + p_2 \cdot x_5(t_f)^2 + p_3 \cdot \dot{x}_2(t_f)^2 \tag{16}$$

The constraint, Eq. (14), will thus be approximately fulfilled at time t_f by this penalty function method. The advantages are that we reduce the number of differential equations integrated backwards in the DDP-algorithm, at the same time as the algorithm keeps its simplicity. The choice of the weight coefficients will be commented on later.

In summary, Eqs. (8) with Eqs. (11-13) and (16) constitute the nonlinear minimizing control problem.

The second formulated tactical requirement is met if we choose

$$V(x_0;t_0) = F(x(t_f);t_f)$$

$$= p_1 \cdot (x_2(t_f) - k \cdot x_1(t_f))^2 + p_2 (x_5(t_f) - \arctan k)^2$$

$$+ p_3 \cdot x_4(t_f)^2 \tag{17}$$

and in this case we let the final time t_f be a free variable. k is the slope of the line. Note, Eq. (17) is formulated in the same way as Eq. (16).

Modified Differential Dynamic Programming Method

A well-known necessary condition for an optimal trajectory, $x^*(t)$, if it exists, is Bellman's partial differential equation

$$V_t(x^*;t) + \min_u H(x^*,u,V_x;t) = 0 \tag{18}$$

where H is the Hamiltonian function defined as

$$H = H(x,u,V_x;t) = L(x,u;t) + V_x^T f(x,u;t) \tag{19}$$

The constraints, Eqs. (9) and (10), have been excluded in the shortened theory in this section.

Expand Eq. (18) to first order by setting $x^* = \bar{x} + \delta x$, then we obtain the additional equations for the first-order DDP method as follows:

$$-\dot{a} = H(\bar{x},u^*,V_x;t) - H(\bar{x},\bar{u},V_x;t) \qquad a(t_f) = 0 \tag{20}$$

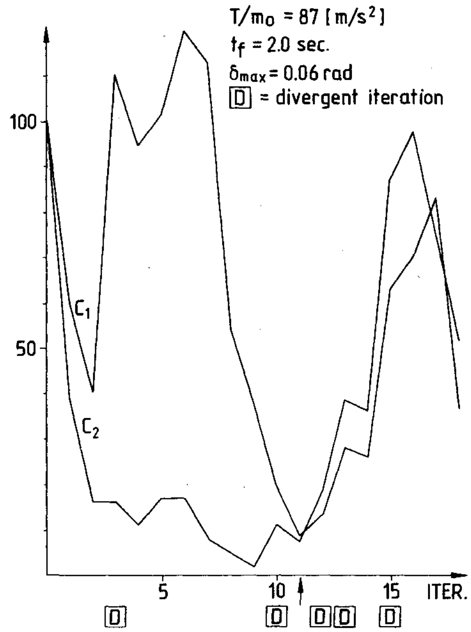

Fig. 5 Convergence control parameters C_1 and C_2 as functions of the iterations.

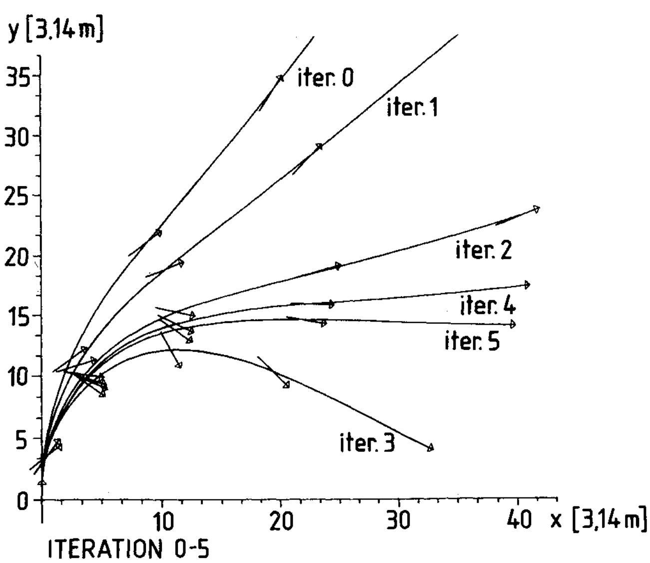

Fig. 6 Convergence sequence in the trajectories: $T/m_0 = 87$ m/s^2, $t_f = 2$ s, $\delta_{max} = 0.06$ rad.

Table 1 Results

t_f, s	T/m_0, m/s^2	δ_{max}, rad	θ_f, rad	$\dot{y}_f$, m/s	y_f, 3.14 m
3.0	130	0.06	0.003	3.0	10.0
	87	0.06	0.001	2.4	13.3
	43	0.06	0.001	1.0	16.3
2.5	130	0.06	0.000	2.5	13.0
	87	0.06	0.005	0.5	13.3
	43	0.06	0.017	0.1	16.7
2.0	130	0.06	0.001	1.0	13.7
	87	0.06	0.003	0.6	15.0
	43	0.06	0.002	...	19.7
1.0	130	0.06	0.003	...	15.7
	87	0.06	0.001	...	13.7
	43	0.06	0.000	...	9.3
3.0	87	0.03	0.015	1.3	19.7
2.5	87	0.03	0.012	4.0	15.3
2.0	87	0.03	0.003	1.0	18.0
1.0	87	0.03	0.002	...	14.3

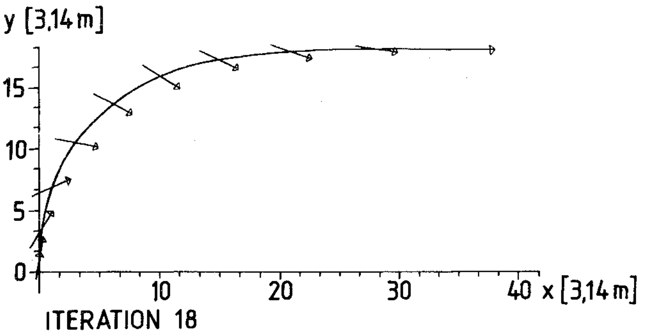

Fig. 7 Optimal trajectory: $T/m_0 = 87$ m/s^2, $t_f = 2$ s, $\delta_{max} = 0.03$ rad.

$$-\dot{V}_x = H_x(\bar{x}, u^*, V_x; t) \qquad V_x(t_f) = F_x(\bar{x}(t_f); t_f) \tag{21}$$

$$u^* = \arg\min_u H(\bar{x}, u, V_x; t)$$

The convergence control is accomplished by introducing a set of Convergence Control Parameters (CCP) as a penalty term added to the integrand in Eq. (7). See Ref. 3. The Hamiltonian, Eq. (19), will then be exchanged for

$$\tilde{H} = H + \tfrac{1}{2}\delta u^T C^i \delta u$$

where $\delta u = u^i - u^{i-1}$, i.e., the difference in optimal control obtained in the present iteration, i, and the previous iteration, $i-1$, and C^i is a diagonal matrix (c^i_{kk}), $k = 1,...,m$. For details of the DDP method see Ref. 2, and for an explanation of the practical management of a DDP algorithm see Ref. 4. In Ref. 3, it is discussed how to make the changes in the convergence control matrix based on information from the predicted and real-cost changes.

Differentiated Convergence Control

When applying the DDP algorithm with the CCP technique to the problem discussed in this paper, one can observe that the control variable $\delta(t)$ has a different influence on the convergence in different subsets of the total time-interval $[0; t_f]$. To improve the convergence properties, one can extend the CCP technique by dividing the control variable into several parts and then use individual convergence control parameters.[5] The predicted cost changes must also be divided into corresponding parts as

$$\Delta a(t_j) = \int_{t_j}^{t_{j+1}} [H(\bar{x}, u^*, V_x; t) - H(\bar{x}, \bar{u}, V_x; t)]\,dt \tag{22}$$

where j is the jth interval of $[0; t_f]$.

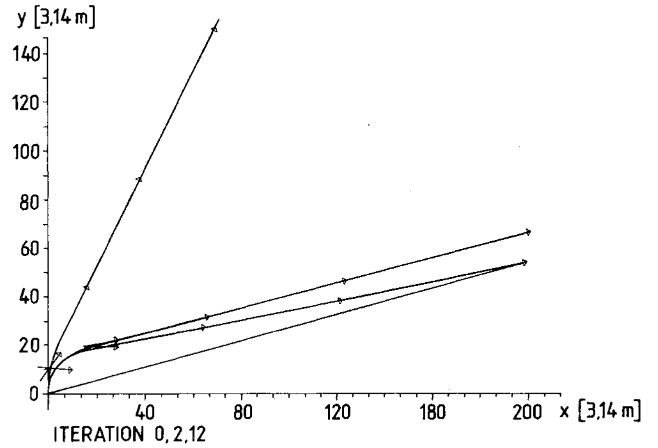

Fig. 8 A convergence sequence toward a given line.

This device has been tested on the problem mentioned above and $\delta(t)$ has been divided into two parts. In the first part, consisting of about 50% of the time interval $[0; t_f]$ where we have the fast changes in $\delta(t)$, we use $\delta_1(t)$ and the corresponding convergence control parameter C_1 and in the remaining time interval, where we have slower dynamics, we use $\delta_2(t)$ and C_2, respectively.

Figure 5 illustrates the case shown in Fig. 6, where C_1 and C_2 have been plotted as functions of the iterations. Thus, C_i constitutes a precautionary step to get the series expansion of Bellman's equation, Eq. (18), valid and to get fast convergence, e.g., a large C_i indicates high precaution.

Results

The program was used to compute the minimal altitude at the final time t_f given that the attitude $\theta = 0$ and the vertical velocity $\dot{y} = 0$ under the constraint that the deflection δ is limited to the value δ_{max}. This was done for different values of the thrust T, producing the results shown in Table 1.

The results indicate that the final altitude depends critically on how the constraints at the final time are achieved. At larger t_f, of course, lower altitudes are reachable with limited control δ and the constraint $\dot{y}_f \approx 0$. The value of the thrust does not seem to be very critical in order to reach low, final altitudes. The δ-limits used (0.06 rad and 0.03 rad, respectively) are very moderate and are achievable with almost all types of deflection arrangement (mechanical, fluid injection).

Some trajectories are shown in Figs. 6-8, where Fig. 6 shows the convergence sequence in the trajectories. Figure 8 shows a convergence sequence in the case where the missile at time t_f has to reach a position on a given line through the launch site with a velocity along this given line. It is probably possible to improve the convergence in this case and also to reach the line closer to the origin. This is done by requiring a smaller t_f.

The weight coefficients p_2 and p_3 in Eq. (16) are chosen to acceptably fulfill the constraints at time t_f and simultaneously the relation to the p_1-value to be such that good convergence is established. Representative values of p_1, p_2, and p_3 are 2×10^{-3}, 3×10^{3}, and 5×10^{-2}. For comparative studies (e.g., different thrust levels) it is important that the parameter values give corresponding contributions from the different terms to the cost functional V.

Concluding Remarks

The computation requirements of the Differential Dynamic Programming (DDP) algorithm are today not attractive for real-time onboard implementations. However, the exact numerical solution of the optimal control problem, obtained in an open-loop form, are among other things compared to suboptimal results from simulations of closed-loop control laws. The trajectories computed show what is possible to achieve with different technical constraints influencing the result. Inclusion of Earth gravity is necessary to reach tactically valid conclusions, but a more complicated model is not believed to be necessary.

The optimization technique using Differential Dynamic Programming with Convergence Control Parameters seems to handle this problem quite easily reaching the optimal trajectory often within five or six iterations. However, it is worth noticing that for sufficiently small δ_{max} we have controls that are suggestive of a "bang-bang" characterization. For larger δ_{max}, this behavior vanishes though the Hamiltonian function still is believed to be approximately linear in δ. The algorithm is approximating the expected behavior of a "bang-bang" solution of very short switching time intervals. However, physically the solutions are considered as quite probable.

Acknowledgments

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